Revista
Ciências Exatas

ISSN: 1516-2893

Vol. 19 | Nº. 2 | Ano 2013

DETERMINATION OF THE COMPLEX POTENTIAL FLOW AROUND TWO PARALLEL AIRFOILS

Jan Mateu Armengol

State University of Campinas

jan@fem.unicamp.br

Carlos Salinas Sedano

State University of Campinas

csalinas@fem.unicamp.br

ABSTRACT

This paper aims to present a methodology to calculate the complex potential flow around two parallel airfoils. For this purpose, the complex potential flow around two circular cylinders is determined based on the Circle Theorem. Thereafter, a double application of the Joukowsky Conformal Transformation transforms the two circular cylinders into the two airfoils. Moreover, the circulation around the airfoils is determined according to the Kutta Condition. To validate the formulation, the stream function over different configurations is presented and then the streamlines identifying the body surfaces are verified.

Keywords: Potential Flow, Complex Variables, Joukowsky Transformation, Airfoil, Circle Theorem.

Correspondência/Contato

UNIVERSIDADE DE TAUBATÉ
Departamento de Engenharia Mecânica

Rua Daniel Danelli, s/n, Jd. Morumbi
Taubaté - SP
CEP 12060-440
Fone (12) 3625-4193

1 INTRODUCTION

The procedure presented in this paper could be divided in two main steps. The initial one of them consists in the determination of the potential flow past two circle cylinders. Regarding this solution, hardly any analytical results exist. Despite Burnside [1], D.G. Crowdy [2] who gets an analytical solution based on the circle domains and Williams [3] who gets the solution based on the work of W. Müller [4] for two lifting circles.

The two circles are then transformed into two parallel airfoils based on of the Joukowsky Conformal Transformation. Once the solution is established, pressure and velocity distributions around the airfoil are easily determined. Also, to calculate the forces experimented on the airfoil due to the flow it could be use, for example, the famous *Balsius formula* for the force acting on two-dimensional body in an inviscid flow.

Therefore, the target of this study is to solve complex potential flow around two parallel airfoils using the method of complex variables, the Joukowsky Transformation and the Circle Theorem. Complex formulation of potential flow is in Warsi [5]. This achievement could be applied, for instance, in general turbo machinery field to study the flow behaviour between compressor's or turbine's blades.

2 MATHEMATICAL FORMULATION

2.1 Method of Complex Variables

This method is very powerful for two-dimensional steady irrotational inviscid and incompressible flows. The complex potential function is defined *as* $w(z)=\phi(x,y)+i\psi(x,y)$,where $i=\sqrt{-1}$ and $z=x+iy$. Thus, both ϕ and ψ, which are the real and imaginary parts of the analytic function w(z), satisfy the Laplace equation. We now define the conjugate complex velocity as $\overline{V}=u-iv$.

The basic linear analytic functions for inviscid flows used in this paper are:

Uniform flow. $w_{flow}(z)=\overline{V}_{\infty}z$, (1)

Flow due to a point vortex. $w_{vor}(z)=\frac{\Gamma}{2\pi i}ln(z)$, (2)

Flow due to a Dipole or Doublet. $w_{dip}(z)=\frac{m}{2\pi z}$, (3)

where, m is the intensity of a dipole, Γ is the circulation, and $\overline{V}_{\infty}$ is the conjugate complex velocity in a point placed in the infinity.

2.2 Flow Past a Circular Cylinder with Circulation at an Angle of Attack [5]

The potential complex function $w_{cil}(z)$ of the flow past a circular cylinder of a radius a, with circulation Γ, and center at the origin, can be obtained as a result of the superposition of three simpler potential complex function. As follows:

$$w_{cil}(z)=w_{flow}(z)+w_{dip}(z)+w_{vor}(z). \quad (4)$$

The null streamline $\psi=0$ of the sum of the equation (1) and (3) must be the equation of a circle of radius a. This condition and the fact that $w_{vor}(z)$ is not considered, amounts to circumstance that the velocity at the surface of the cylinder must be everywhere tangential since the velocity normal to a stationary cylinder must be zero (As it is well known, the lines which ψ are constant due to a vortex are concentric circles around the its center). Besides y = 0, the other null streamline is obtained if: $x^2+y^2=a^2$. And then, $m=2\pi a^2 u_{\infty}$.

Thus, the complex potential for the flow past a stationary circular cylinder with circulation placed on a uniform stream is:

$$w_{cil}(z)=\overline{V}_{\infty}z+\frac{V_{\infty}a^2}{z}-\frac{\Gamma}{2\pi i}\ln(z), \quad (5)$$

this is a well known solution, and it has obvious application to basic problems in aerodynamics.

The complex conjugate velocity is then:

$$\overline{V}=\frac{dW}{dz} \quad (6)$$

2. 3 Flow Past two Circular Cylinders at an Angle of Attack

The potential complex function describing the flow past two circular cylinders is determined base on the work of Williams [3]. Thus, it has been calculated as the superposition of two components: a *streaming flow* past both circles and a *circulation flow*.

The potential function of the *streaming flow* past both circle is founded in the Circle Theorem [6], which states that if a circle $|\varsigma| = a$ is introduced into a flow, represented by the complex potential, $w = f(\varsigma)$, then the complex potential becomes:

$$w = f(\varsigma) + \bar{f}\left(\frac{a^2}{\varsigma}\right). \tag{7}$$

According to the Circle Theorem, a second cylinder is introduced in $f(\varsigma)$, when $f(\varsigma)$ is representing a flow past a circular cylinder. Nonetheless, due to the introduction of the second cylinder, the first one becomes modified. So, it is no more represented by a streamline. The Circle Theorem is again applied to all the singularities outside the first circle. This produces two new doublet images, which are the reflections of the images produced in the second circle by the previous step.

Thus, this process is repeated and after each step, either the first or the second circle is a streamline. Each reflection entails the addition of two image doublets. The strength of the doublets, which are added after each reflection, is monotonic decreasing and approaches zero. After several reflections, the new image doublets will only slightly change the complex potential of the system. This establishes a necessary condition for the series converging to the complex potential for the streaming flow past two circles. The proof of a sufficient condition for convergence is given in Williams [3].

In the case of this paper, the potential flow past two cylinders has been calculated applying the Circle Theorem five times, placing this way nine different dipoles. The potential complex results:

$$w_{flow}(\varsigma)=\bar{V}_\infty\varsigma+\frac{V_\infty a^2}{\varsigma}+\frac{V_\infty b^2}{(\varsigma-f)}-\frac{a^2b^2\bar{V}_\infty}{f^2\left(\varsigma-\frac{a^2}{f}\right)}+\frac{a^4b^2V_\infty}{f^2\left(f-\frac{b^2}{f}\right)^2\left(\varsigma-\frac{a^2}{f-\frac{b^2}{f}}\right)}-\frac{b^4a^4\bar{V}_\infty}{f^2\left(f-\frac{a^2}{f}\right)^2\left(f-\frac{b^2}{f-\frac{a^2}{f}}\right)^2\left(\varsigma-\frac{a^2}{f-\frac{b^2}{f-\frac{a^2}{f}}}\right)}$$

$$+\frac{a^6b^4V_\infty}{f^2\left(f-\frac{b^2}{f}\right)^2\left(f-\frac{a^2}{f-\frac{b^2}{f}}\right)^2\left(f-\frac{b^2}{f-\frac{a^2}{f-\frac{b^2}{f}}}\right)^2\left(\varsigma-\frac{a^2}{f-\frac{b^2}{f-\frac{a^2}{f-\frac{b^2}{f}}}}\right)}-\frac{a^2b^2\bar{V}_\infty}{f^2\left(\varsigma-\left(f-\frac{b^2}{f}\right)\right)}+\frac{b^4a^2V_\infty}{f^2\left(f-\frac{a^2}{f}\right)^2\left(\varsigma-\left(f-\frac{b^2}{f-\frac{a^2}{f}}\right)\right)}-\frac{a^4b^4\bar{V}_\infty}{f^2\left(f-\frac{b^2}{f}\right)^2\left(f-\frac{a^2}{f-\frac{b^2}{f}}\right)^2\left(\varsigma-\left(f-\frac{b^2}{f-\frac{a^2}{f-\frac{b^2}{f}}}\right)\right)} \tag{8}$$

where a is the radius of the circle placed on the origin and b is the radius of the second cylinder placed on a distance f from the origin. The solution for the first circle is exact, what means that the surface of the circle is perfectly matched with a streamline, while the error committed in the second cylinder with the values adopted in this paper is less than 10^{-3}.

In order to define the *circulation flow* with Γ_1 circulation around the first circle is constructed by placing a vortex in its centre. According to the previous case, the second circle is introduced in the flow using the Circle Theorem. This process is repeated, alternatively making the first and second circle streamlines, each reflection effectively entails the addition of two vortices of opposite sense at inverse points in one of the circles. So, the resultant circulation produced by adding two vortices of opposite sense becomes null. A proof of sufficiency is given in Williams [3]. In the example presented here, the Circle Theorem has been applied two times, so it has been involved five different vortexes. It follows bellow the potential complex function for the circulation:

$$w_{circ}(\varsigma)=-\frac{\Gamma_1}{2\pi i}\ln(\varsigma)-\frac{\Gamma_1}{2\pi i}\ln(\varsigma-f)+\frac{\Gamma_1}{2\pi i}\ln\left(\varsigma-\left(f-\frac{b^2}{f}\right)\right)+\frac{\Gamma_1}{2\pi i}\ln\left(\varsigma-\frac{a^2}{f}\right)-\frac{\Gamma_1}{2\pi i}\ln\left(\varsigma-\frac{a^2}{f-\frac{b^2}{f}}\right), \tag{9}$$

The same proceed has been done for the second circle in order to provide a Γ_2 circulation around it. Obtaining this way, the potential complex function for the circulation around the second circle, $w_{circ2}(\varsigma)$.

So, the total complex potential flow past both circular cylinders with circulation is calculated as a superposition of the three components below:

$$w_{total}(\varsigma)=w_{flow}(\varsigma)+w_{circ1}(\varsigma)+w_{circ2}(\varsigma). \quad (10)$$

In order to determine the amount of circulation which actually exists in this situation, an application of the following postulate known as the Kutta condition as proposed.

Kutta condition declares: "Out of the infinite number of possible inviscid potential flows past an airfoil with sharp trailing edge, the flow that physically occurs is the one in which there is no velocity discontinuity at the trailing edge." This implies that there is a smooth flow with finite velocity past the trailing edge. This unique case of circulation, as is well known, occurs when the point on the circle in the *ς-plane* which corresponds to the trailing edge in the *z-plane* is a stagnation point, a mathematical demonstration of this can be found in Warsi [5]. Due to the symmetry of the airfoil, the points on the circles which corresponds to the trailing edge of the airfoils are in the points *(-a,0)* for the first circle, and *(f-b,0)* for the second circle. So, the circulations, Γ_1 and Γ_2, can be determined with the following equation system.

$$\frac{dWt(-a)}{d\varsigma} = 0 \,, \frac{dWt(f-b)}{d\varsigma} = 0 \,. \quad (11)$$

2.4 Joukowsky Mapping

The Joukowsky transformation is utilized to transform the two circle in the *ς-plane* in an airfoil and a quasi circle in the *z-plane*. The equation utilized for the transformation is a particular case which can transform a circle placed on the origin into x-symmetrical and y-asymmetrical airfoil. This is:

$$z=\varsigma+\epsilon+\frac{c^2}{\varsigma+\epsilon}\,, \quad (12)$$

where c and ϵ are real parameters for the geometrical characterization of the airfoil.

While the first circle, placed on the origin, is transformed into an airfoil, the transformation of the second one can be considered as a circle. Consider g the point placed at an angle of $\pi/2$ of the quasi circle transformed, the quasi circle transformed is approximated to a circle with the following equation:

$$Im(g)e^{i\alpha}+Re(g)\,,\ 0<\alpha<2\pi\,, \quad (13)$$

where,

$$g=\frac{-b^2+c^2+2ib(f+\epsilon)+(f+\epsilon)^2}{ib+f+\epsilon}\quad . \quad (14)$$

In this consideration, it is been committed an error less than 10^{-4} for the taken values in this paper. The error has been calculated as the maximum difference between the radius of the quasi circle transformed and the radius of the circle presented with equation (13) above.

Then, the Joukowsky transformation is applied again in order to get a set of two airfoils. Now, the modified Joukowsky transformation defined below place the quasi circle on the origin in order to transform it into the desired second airfoil.

$$z_2= z+\epsilon-s+\frac{c^2}{z+\epsilon-s} \quad (15)$$

Where s is the image of the origin of the second transformation due to the first Joukowsky transformation, therefore, $s=f+\epsilon\frac{c^2}{f+\epsilon-}$. Note that consequently the airfoil due to the first circle is also displaced in a distance s on the negative real axe.

3 RESULTS

Taking a complex uniform flow velocity V_∞ = -5 + 0.5i m/s, placing the circle of radius a = 1.4m in the origin and the other circle of radius b = 1.43m in a distance f = 6.5m of the origin in the positive real axis. The streamlines obtained are:

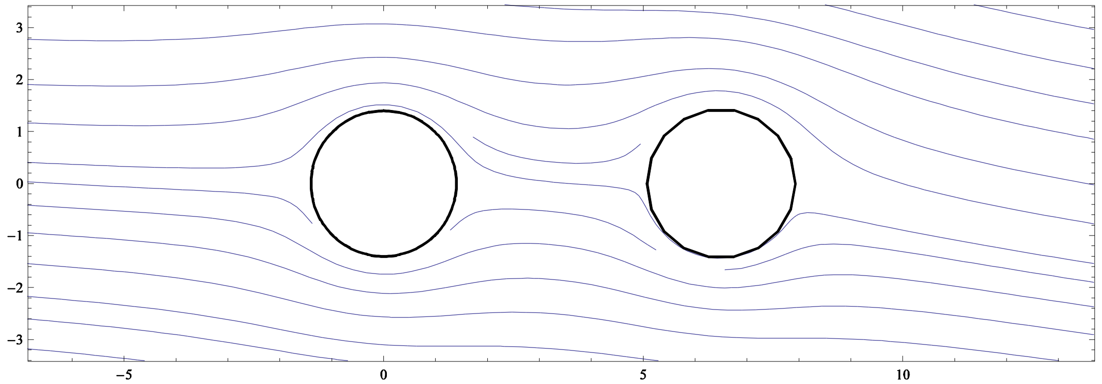

Figure 1: Streamlines for a flow past two circular cylinders with circulation at an Angle of Attack

Note that the radius *b* of the second circle has been taken slightly greater than the radius *a* in order to compensate the deformation due to the first Joukowsky transformation, according to the equation (13). That is to obtain two identical airfoils in the final configuration.

Now proceeding with the double application of the transformation: the streamlines are obtained adjusting the base circle of the transformation $c = 1m$, and the horizontal displacement ε=0.2 *m*. Also, according to the Kutta Condition, the circulation around the airfoils are $\Gamma_1 = -4.897 m^2/s$ and $\Gamma_2 = 12.051\ m^2/s$,

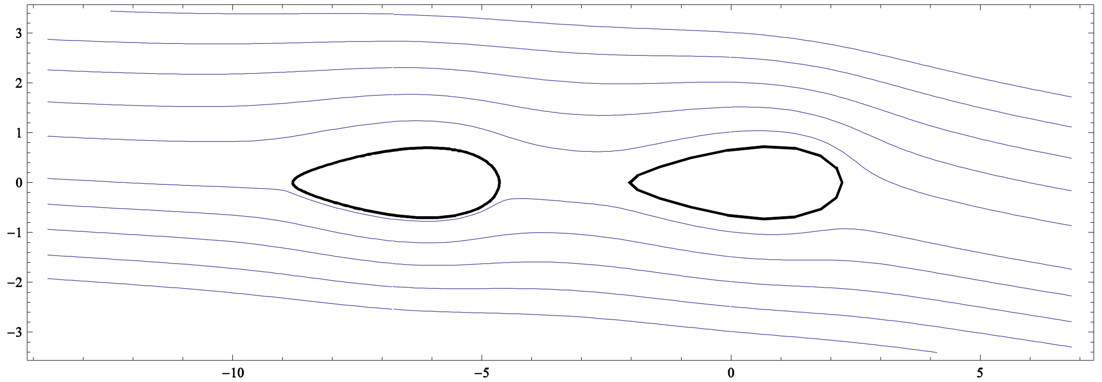

Figure 2: Streamlines for a flow past two parallel airfoils at an Angle of Attack

In order to validate the formulation, the stream function on different configurations of airfoils is calculated then the streamlines identifying the airfoils surfaces are verified. Also, the velocity field is checked over the airfoil configuration set, especially the continuity in the trailing edge. The symmetry for the case with no circulation in the airfoils and horizontally flow are also checked.

The opportunity has been taken to examine the behaviour of solutions with large and small distances between the airfoils. At large separations, one aerofoil does not influence the flow around the second aerofoil, thus the result is trivial. However it provided a useful check of the solution by comparison with the solution of the single airfoil theory and these are used as a proof of the given method. In a small separation the quasi circle is more deformed as a result of the first application of the transformation so the approximation becomes less accurate.

It follows below two graphics in order to front the influence of the circulation in the stagnations point at the trailing edge. Figure 3 shows the streamlines of the potential flow which was determined by taking the circulation as null, so the stagnation point is not localized in the trailing edge.

In the other hand, figure 4, shows the streamlines of the potential flow when the amount of circulation is calculated according to the Kutta condition. As a result of this, the stagnation point is displaced to the trailing edge.

4 CONCLUSIONS

The main achievement of the procedure presented in this paper consists in a simply formulation which results describe the potential flow past the two airfoil's contour. Due to the characteristics of the Circle Theorem, one of the airfoils will be defined by a streamline. On the other hand, the contour of the other airfoil will be a good approach of the analytical solution. However, the numerical approximation can be made as close to the exact value, as is desired, by increasing the number of terms in the truncated part of the series.

This process allows to calculate the pressure and velocity distribution; and the streamlines arrangement. The formulation has been made such that is possible to get the solution flow with different characteristics of the flow (angle of attack, velocity modulus, pressure and density), different geometry of the airfoils and also different distance between them.

The potential flow is an approximation of a real physic flow; nevertheless it has enormous advantages in front of other viscid methods. The first main favoring circumstance to use this method resides in its simplicity. The second one, lies on the fact that the result of this method is a continue function extended in the entire *z-plane*, which is able to set the velocity and pressure of any point on it. Furthermore, the mathematical problem solved here is a basic one in potential theory and the solutions can be applied to many other branches of physics. This simple but powerful formulation is expected to be applied in a next work in the iced airfoil aerodynamics field.

ACKNOWLEDGMENTS

The authors wish to thank the CNPQ for the financial support.

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