AN EXTENDED VARIATIONAL PRINCIPLE FOR NON-LINEAR AND NON-POTENTIAL POTENTIALIZABLE OPERATORS

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ABSTRACT

For non-conservative mechanical systems (non-potential operators), classical energetic variational principles do not hold true. In this article, a generalized variational principle, valid for non-linear and non-conservative systems is deduced by means of "potentializable" operators. A systematic inclusion of boundary conditions in problems dealing with such operators is proposed and examples from continuum mechanics are presented.

KEY-WORDS: Variational principles, non-linear mechanics, non-conservative mechanical systems

INTRODUCTION

Historic background

The variational principles are presently used mainly for:

i) deriving the differential equations for a physical problem along with their correspondent boundary conditions;
ii) studying the symmetry and conservation laws under infinitesimal transformation groups;
iii) proving that a given boundary problem is solvable (i.e., demonstrating of the existence of solutions for non-linear equations);
iv) solving linear and non-linear problems by means of direct methods of variational

$$\vec{F} = \frac{\partial V}{\partial \vec{r}} = -\nabla V$$

calculus.

The use of variational principles in cases iii and iv leads to the so-called "inverse problem of the calculus of variations", i.e., the existence and formulation of a functional, whose variation, once vanished, supplies the boundary problem in consideration.

Historically the inverse problem of the calculus

$$\vec{F} = \frac{\partial V}{\partial \vec{r}} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{\vec{r}}}\right)$$

of variations is related to the idea of finding a potential to a non-conservative force.

The first work in this context, due to Lagrange (1773), conceives as conservative any force, which derives from a scalar potential V, function of position only:

However, not all forces are potential in Lagrangean sense.

A first attempt to widen the applicability of the Lagrangean concept of potential, comes with

$$\frac{\partial F_i}{\partial r_j} = \frac{\partial F_j}{\partial r_i}$$

Weber (1848), who tried to generalize the idea of a potential V supposing that it could depend not only on position but also on velocity of the particles in analysis. In this case the force is defined by:

The condition for $\vec{F}$ to derive from a potential is that the following symmetry relationship between the force $\vec{F}$ and the position vector $\vec{r}$ must stand:

Helmholtz (1886) derived such conditions for potential forces as those proposed by Weber.

As one can observe the definition due to Weber is equivalent to state that $\vec{F}$ shall be equal to the variational derivative of a potential scalar function V when it depends just on position and velocity.

The generalization of these ideas comes with the definition of a potential operator as an operator acting from a Hilbert space to its dual that can be obtained from the gradient of a functional for all elements of its domain.

The symmetry and potentiality conditions for operators, first enunciated by Kerner (1933), were demonstrated by Vainberg (1973), relating the symmetry of an operator to the equality of its second-order Gâteaux derivatives for all elements of its domain and the potentiality to this symmetry and the continuity of those Gâteaux derivatives.

It is observed, however, that the referred condition restricts the applicability of the strong (or energetic) variational formulation to the class of operators symmetric according to Kerner.

The main goal of this work is to extend this class of operators, to a class that will be called potentializable.

In order to achieve it, a generalized variational principle (applied to several fields) is deducted, aiming to an energetic variational approach of boundary value problems that includes operators neither linear nor potential.

A variational formulation applied to more than one field is known as a "Mixed Variational Principle".

The use of such principles, in Mechanics of Solids, started with the works of Reissner (1948, 1953) who presented a principle for the theory of elasticity based on the variations of displacements and stresses. Reissner by his turn, relied on some ideas previously established by Hellinger (1914) completing them through inclusion of boundary terms.

Following those basic ideas many generalizations appeared. Among them one should mention Hu and Washizu (1955, 1955), Prager (1967), Frejis de Veubeke (1964) and Tonti (1969).

The concept of complementary principle, after some ideas of Legendre, was introduced in Mathematical Physics by means of the notion of "Hypercicle" due to Prager and Synge (1947, 1957).

Noble (1963) generalizes that concept and Sewell (1969, 1973) applies systematically the Legendre's ideas to the elaboration of complementary variational principles in Mechanics. Various refinements of this theory followed through the works of Arthurs (1970), Robinson (1971), Oden and Reddy (1972, 1973, 1975, and 1976) and Reddy (1973, 1976).

In this article, a systematic inclusion of boundary conditions in problems dealing with potentializable operators is proposed and particular cases, using Reissner's and quadratic functionals are presented.

Preliminaries

Let H and G be Hilbert spaces (equal or distinct) of the functions $u=u(x)$ defined in a smooth open limited subset $\Omega \subset R^n$ with smooth boundary $\partial\Omega$, with ∂H and ∂G standing for the respective Hilbert spaces of the functions $u = u_0(s)$ defined on $\partial\Omega$.

OPERATORS A AND $A_U^{'}$

Let now H, ∂H and G, ∂G, be the factor spaces of the following Cartesian products:

$$U = H \times \partial H$$

$$V = G \times \partial G$$

and a non-linear operator

$$A: U \to V$$

whose Gâteaux derivative is $A_U^{'} \in L(U,V)$, the set of the linear operators from U to V.

Consider yet that A acts upon U through two operators, F and B_0 with $F: H \to G$, being an operator non-necessarily potential nor linear, and $B_0 : \partial H \to \partial G$, a linear operator.

In such a context A can be represented by:

$$A = \begin{bmatrix} F & 0 \\ 0 & B_0 \end{bmatrix}$$

It can be easily demonstrated that deriving A partially with respect to the U factor spaces in the point $U=(u,u_0)\in \mathbf{U}$, one obtains:

$$A_U^{'}=\begin{bmatrix}[F_u^{'}] & 0\\ 0 & B_0\end{bmatrix}$$

where $[F_u^{'}]$ is the Gâteaux derivative of F in the point $u\in \mathbf{H}$.

TRACE OPERATORS

Suppose that there exist two operators, γ_0 and δ_0, linear, continuous and surjective given by:

$$\gamma_0 : \mathbf{H} \to \partial \mathbf{H} \text{ such that } \gamma_0\, u = u_0$$

$$\delta_0 : \mathbf{G}' \to \partial \mathbf{G}' \text{ such that } \delta_0\, \sigma^{'} = \sigma_0^{'}$$

where G' is the dual space of G.

NOTATION

For the elements of the previously presented spaces, the following symbols are used:

U elements: capital Latin letters

V elements: capital Greek letters

H elements: lower-case Latin letters

G elements: lower-case Greek letters

Boundary elements: subscript 0

Dual spaces' elements: prime (')

(Operators, their Gâteaux derivatives):

$$(A,A_U^{'}),(F,F_u^{'}),(B_0,B_0),(\gamma_0,\gamma_0),(\delta_0,\delta_0)$$

ADJOINT OF $A_U^{'}$

$A_U^{'*}$ indicates the formal adjoint of $A_U^{'}$. As $A_U^{'}\in \mathbf{L}(\mathbf{U},\mathbf{V})$ one has $A_U^{'*}\in \mathbf{L}(\mathbf{V}',\mathbf{U}')$ and

$$\left\langle \Sigma', A_U^{'}V\right\rangle_{\mathbf{V}} = \left\langle A_U^{'*}\Sigma', V\right\rangle_{\mathbf{U}}$$

where $\langle .,.\rangle_{\mathbf{V}}$ and $\langle .,.\rangle_{\mathbf{U}}$ are the dualities between V and $\mathbf{V}'$ (U and $\mathbf{U}'$), $\forall V\in \mathbf{U}$, $\forall \Sigma'\in \mathbf{V}'$.

These dualities can be expressed in their respective factor spaces, with the substitution of V by $(v,\gamma_0 v)$ and Σ' by $(\sigma',\delta_0\sigma')$, through the following expressions:

$$\left\langle \Sigma', A_U^{'}V\right\rangle_{\mathbf{V}} = \left\langle \sigma',[F_u^{'}][v]\right\rangle_{\mathbf{G}} + \left\langle \delta_0\sigma', B_0\gamma_0 v\right\rangle_{\partial\mathbf{G}}$$

$$\left\langle A_U^{'*}\Sigma', V\right\rangle_{\mathbf{U}} = \left\langle [F_U^{'*}]\sigma', v\right\rangle_{\mathbf{H}} + \left\langle B_0^{*}\delta_0\sigma', \gamma_0 v\right\rangle_{\partial\mathbf{H}}$$

From the equality between these dualities, one can write the following Green's generalized formula:

$$\left\langle \sigma',[F_u^{'}][v]\right\rangle_{\mathbf{G}} = \left\langle [F_U^{'*}]\sigma', v\right\rangle_{\mathbf{H}} + B(\sigma',v)$$

with the bilinear boundary form:

$$B(\sigma',v) = \left\langle B_0^{*}\delta_0\sigma', \gamma_0 v\right\rangle_{\partial\mathbf{H}} - \left\langle \delta_0\sigma', B_0\gamma_0 v\right\rangle_{\partial\mathbf{G}}$$

One should also observe that, due to the surjectivity of γ_0 and δ_0, one has:

$$B(\sigma',v) = B_1(\sigma',v) - B_2(\sigma',v)$$

with

$$B_1(\sigma',\hat{v}) = 0,\ \forall \sigma'\in \mathbf{G}' \Rightarrow \hat{v} = 0$$

and

$$B_2(\hat{\sigma}',v) = 0,\ \forall v'\in \mathbf{G}' \Rightarrow \hat{\sigma}' = 0$$

i.e., $B(\sigma',v)$ is a non degenerated form.

CANONICAL INJECTION

Let $J : V \to V'$ be the canonical injection from V to V'. In this case, for each $\Sigma \in V$ one has a $\Sigma' \in V'$ given by $\Sigma' = J\Sigma$, or, showing inner and boundary elements $\Sigma' = (\sigma', \sigma_0') = (J\sigma, J\sigma_0) = J\Sigma$

Figure 1 shows the operators, spaces and variables presented in this chapter.

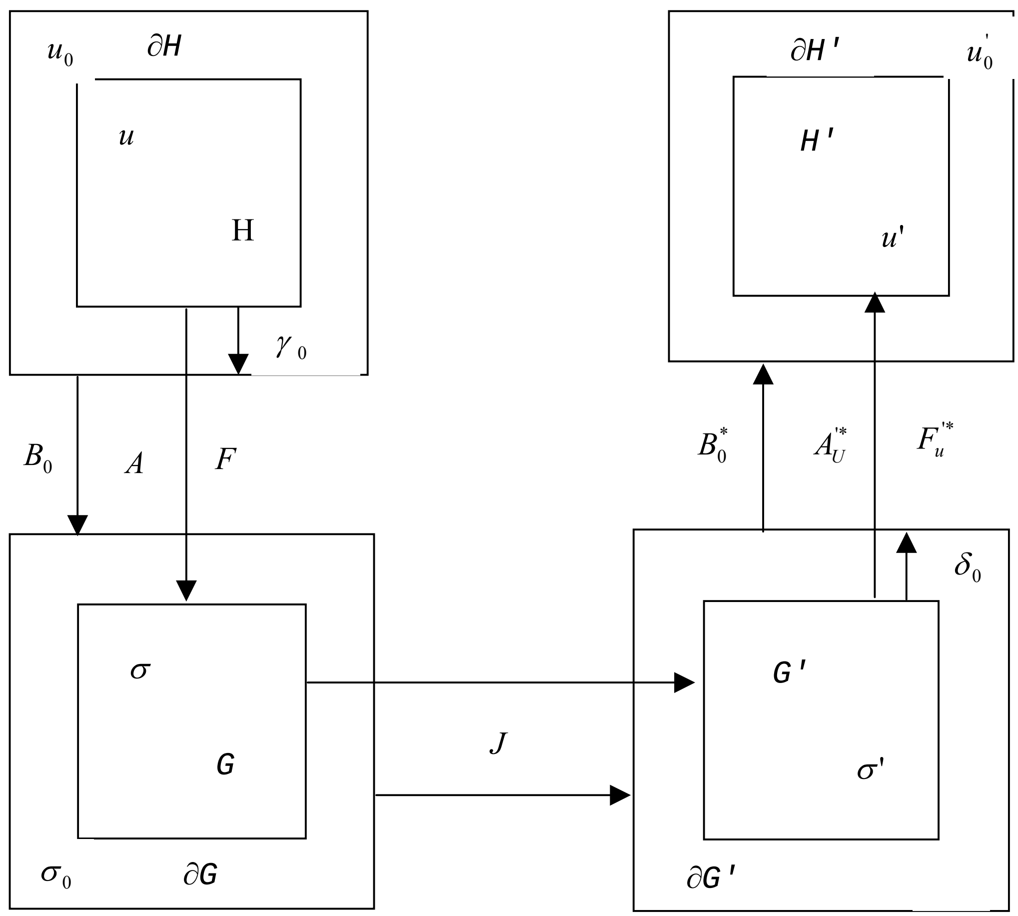

Figure 1 - Spaces, variables and operators used in this article

FUNCTIONAL $f(U, \Sigma, \Sigma')$

Consider the functional:

$$f(U, \Sigma, \Sigma') = \frac{1}{2}\langle J\Sigma, \Sigma \rangle_V + \langle \Sigma', AU - \Sigma \rangle_V - \langle U', U \rangle_U$$

It is shown hereafter that searching the critical point of this functional is equivalent to the search of the solution of the following boundary-value problem:

$$AU = \Sigma \text{ in } V$$

$$J\Sigma = \Sigma' \text{ in } V'$$

$$A_U'^{*}\Sigma' = U' \text{ in } U'$$

where:

$$U = (u, \gamma_0 u) \quad , \quad \Sigma = (\sigma, \sigma_0)$$
$$\Sigma' = (\sigma', \delta_0 \sigma') \ , \quad U' = (u', u_0')$$

More explicitly one has that the critical point of the functional *f* corresponds to the problem:

$$Fu = \sigma \qquad \text{in } \Omega$$
$$J\sigma = \sigma' \qquad \text{in } \Omega$$
$$[F_u^*][\sigma'] = u' \qquad \text{in } \Omega$$
$$B_0 \gamma_0 u = \sigma_0 \qquad \text{on } \partial\Omega$$
$$J\sigma_0 = \sigma_0' \qquad \text{on } \partial\Omega$$
$$B_0^* \delta_0 \sigma' = u_0' \qquad \text{on } \partial\Omega$$

DEMONSTRATION

Indicating as $\delta U, \delta\Sigma$ and $\delta\Sigma'$ the increments in U, Σ and Σ', respectively, one has for

$$f(U, \Sigma, \Sigma') = \frac{1}{2}\langle J\Sigma, \Sigma\rangle_V + \langle \Sigma', AU - \Sigma\rangle_V - \langle U', U\rangle_U$$

that:

$$\delta f(U, \Sigma, \Sigma', \delta U, \delta\Sigma, \delta\Sigma') = [\langle J\Sigma, \delta\Sigma\rangle_V + \langle \Sigma', -\delta\Sigma\rangle_V] + [\langle \delta\Sigma', AU - \Sigma\rangle_V] + [\langle \Sigma', A_U' \delta U\rangle_V - \langle U', \delta U\rangle_U]$$

where within brackets one varied, in order, Σ, Σ' and *U*.

Taking into account that $\langle \Sigma', A_U' \delta U\rangle_V = \langle A_U'^{*}\Sigma', \delta U\rangle_U$ the variation of *f* results:

$$\delta f(U, \Sigma, \Sigma', \delta U, \delta\Sigma, \delta\Sigma') = [\langle J\Sigma - \Sigma', \delta\Sigma\rangle_V] + [\langle \delta\Sigma', AU - \Sigma\rangle_V] + [\langle A_U'^{*}\Sigma' - U', \delta U\rangle_U] \, .$$

Representing now U, Σ, Σ', U' by their components $(u, \gamma_0 u), (\sigma, \sigma_0), (\sigma', \delta_0 \sigma'), (u', u_0')$ in their respective dualities one obtains:

$$\langle J\Sigma - \Sigma', \delta\Sigma\rangle_V = \langle J\sigma - \sigma', \delta\sigma\rangle_G + \langle J\sigma_0 - \sigma_0', \delta\sigma_0\rangle_{\partial G}$$
$$\langle \delta\Sigma', AU - \Sigma\rangle_V = \langle \delta\sigma', Fu - \sigma\rangle_G + \langle \delta_0 \delta\sigma', B_0 \gamma_0 u - \sigma_0\rangle_{\partial G}$$

$$\langle A_U'^{*}\Sigma' - U', \delta U\rangle_U = \langle [F_u^*][\sigma'] - u', \delta u\rangle_H + \langle B_u^* \delta_0 \sigma' - u_0', \gamma_0 \delta u\rangle_{\partial H}$$

Imposing $\delta f = 0$ the previously announced system of equations is obtained.

It should be observed, in addition, that in terms of factor spaces:

$$f(u, \sigma, \sigma') = \frac{1}{2}\langle J\sigma, \sigma\rangle_G + \langle \sigma', Fu - \sigma\rangle_G + \frac{1}{2}\langle J\sigma_0, \sigma_0\rangle_{\partial G} - \langle u', u\rangle_H + \langle \delta_0 \sigma', B_0 \gamma_0 u - \sigma_0\rangle_{\partial G} - \langle u_0', \gamma_0 u\rangle_{\partial H}$$

PARTICULAR CASES OF THE FUNCTIONAL $f(u, \sigma, \sigma')$

Reissner functional $R(u, \sigma)$

If one restrains, a priori, Σ' to the subspace V by means of $\Sigma' = J\Sigma,\ \forall \Sigma \in V$, then the functional $f(u,\sigma,\sigma')$ becomes:

$$R(u,\sigma) = f(u,\sigma,J\sigma) = \frac{1}{2}\langle J\sigma,\sigma\rangle_G + \langle J\sigma, Fu-\sigma\rangle_G + \\ + \frac{1}{2}\langle J\sigma_0,\sigma_0\rangle_{\partial G} - \langle u',u\rangle_H + \\ + \langle \delta_0 J\sigma, B_0\gamma_0 u - \sigma_0\rangle_{\partial G} - \langle u_0^{'}, \gamma_0 u\rangle_{\partial H}$$

The null variation of $R(u,\sigma)$ results in the system:

$$Fu = \sigma \qquad \text{in } \Omega$$

$$[F_u^*][J\sigma] = u' \qquad \text{in } \Omega$$

$$B_0\gamma_0 u = \sigma_0 \qquad \text{on } \partial\Omega$$

$$B_0^*\delta_0 J\sigma = u_0^{'} \qquad \text{on } \partial\Omega$$

Quadratic functional $Q(u)$

Suppose now that a given system is described by:

$$Fu = \sigma \qquad \text{in } \Omega$$

$$JFu = \sigma' \qquad \text{in } \Omega$$

$$B_c u = \sigma_0 \qquad \text{on } \partial\Omega$$

where B_c is an operator that acts from H to ∂G.

The functional associated with this system is:

$$Q(u) = f(u, Fu, JFu) = \frac{1}{2}\langle JFu, Fu\rangle_G - \langle u', u\rangle_H - \\ - \langle u_0^{'}, \gamma_0 u\rangle_{\partial H}$$

The variation $\delta Q(u,\delta u) = 0$ implies:

$$[F_u^*][JFu] = u' \qquad \text{in } \Omega$$

$$[B_c^*][JFu] = u_0^{'} \qquad \text{on } \partial\Omega$$

If one takes G such that J is the identity operator one obtains:

$$Q(u) = \frac{1}{2}\langle Fu, Fu\rangle_G - \langle u', u\rangle_H - \langle u_0^{'}, \gamma_0 u\rangle_{\partial H}$$

and the associated system is reduced to:

$$[F_u^*][Fu] = u' \qquad \text{in } \Omega$$

$$[B_c^*][Fu] = u_0^{'} \qquad \text{on } \partial\Omega$$

APPLICATION OF A REISSNER'S TYPE FUNCTIONAL: BEAM IN BENDING AND COMPRESSION

Introduction

Consider a beam having its neutral axis aligned with the *x*-axis and undergoing simultaneously to bending and compression loads. Its displacements will be considered *u* in *x* direction and *w* in *z*, the transversal (vertical) direction. Its longitudinal dimension is l and its rectangular cross-section dimensions are *b* (in *y* direction) and *h* (in *z* direction).

Variables

This problem can be described in terms of the internal loads and the beam strain. The strain ε is defined by:

$$\varepsilon = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2} = \varepsilon_{xx}$$

or $$\varepsilon = \eta + z\mathrm{N}$$

with

$$\mathrm{N} = -w''$$

and

$$\eta = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

The internal loads are the axial force *N* and the bending moment *M* being defined as

$$N = \int_0^h \tau \, dz$$

$$M = \int_0^h \tau \, z \, dz$$

with $\tau = \tau_{xx}$.

Canonical equilibrium equations

The canonical equilibrium equations are

$$\frac{\partial N}{\partial x} = 0 \qquad (1)$$

$$\frac{\partial^2 M}{\partial x^2} + \frac{\partial}{\partial x}\left(N\frac{\partial w}{\partial x}\right) + q = 0 \qquad (2)$$

subjected to the following boundary conditions:

$$N + \hat{N} = 0 \qquad \text{on}\,\{0, l\}$$

$$Nw' + M' - \hat{Q} = 0 \qquad \text{on}\,\{0, l\}$$

$$N - \hat{M} = 0 \qquad \text{on}\,\{0, l\}$$

being $\hat{N}, \hat{M}$ and $\hat{Q}$, respectively, the axial force, the bending moment and the shear force prescribed on the boundary.

Constitutive relations

The equations relating the internal loads and the terms of the strain are:

$$M = -Eiw'' = EI\mathrm{N}$$

$$N = C\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] = C\eta$$

where $C = Eh$.

Generalised variational principle application

The generalised variational principle, previously presented in this article, can be applied to this problem. Following the notation adopted in the text the intervening entities can now be defined:

A vector $\vec{u}$: $\vec{u} = [u, w]$

An operator $F[\vec{u}]$: $F[\vec{u}] = [\eta, \mathrm{N}] = \sigma$

An element σ': $\sigma' = J\sigma = [C\eta, EI\mathrm{N}]^T$

A derivative of $F[\vec{u}]$:

$$F_u' = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial w \partial}{\partial x \partial x} \\ 0 & -\frac{\partial^2}{\partial x^2} \end{bmatrix}$$

An adjoint of F':

$$F_u^{'*} = \begin{bmatrix} -\dfrac{\partial}{\partial x} & 0 \\ -\dfrac{\partial w'}{\partial x} & -\dfrac{\partial^2}{\partial x^2} \end{bmatrix}$$

The canonical equilibrium equations:

$$F_u^{'*}(\sigma') = \vec{u}'$$

$$F_u^{'*}(F[\vec{u}]) = \vec{u}' = [0, p]^T$$

i.e., $$\begin{bmatrix} -N' \\ -(Nw') - M \end{bmatrix} = \begin{bmatrix} 0 \\ p \end{bmatrix}$$

Now the scheme depicted in Figure 1, with respect to its inner elements, can be visualised:

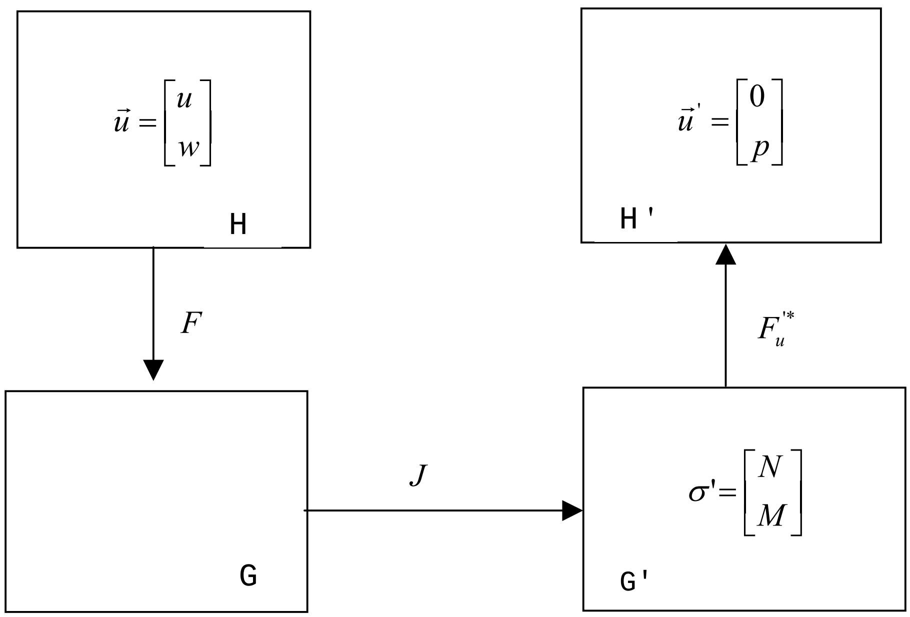

Figure 2 Spaces, variables and operators used in this example

Functional $R(\vec{u}, \sigma)$

According to the previous development a functional, which once varied, will lead to the canonical equilibrium equations with the proper boundary conditions is:

$$R(\vec{u},\sigma)=\langle J\sigma,F(\vec{u})\rangle=\langle J\sigma,F(\vec{u})\rangle_{\mathrm{G}}-\frac{1}{2}\langle J\sigma,\sigma\rangle_{\mathrm{G}}-\langle\vec{u}',\vec{u}\rangle_{\mathrm{H}}+\frac{1}{2}\langle J\sigma_0,\sigma_0\rangle_{\partial\mathrm{G}}+\langle\delta_0 J\sigma,B_0\gamma_0 u-\sigma_0\rangle_{\partial\mathrm{G}}-\langle u_0',\gamma_0 u\rangle_{\partial\mathrm{H}}$$

$$=\left\langle\begin{bmatrix}N\\M\end{bmatrix},\begin{bmatrix}\dfrac{\partial u}{\partial x}+\dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2\\-w''\end{bmatrix}\right\rangle_{\mathrm{G}}-\frac{1}{2}\left\langle\begin{bmatrix}N\\M\end{bmatrix},\begin{bmatrix}\eta\\\mathrm{N}\end{bmatrix}\right\rangle_{\mathrm{G}}-\left\langle\begin{bmatrix}0\\p\end{bmatrix},\begin{bmatrix}u\\w\end{bmatrix}\right\rangle_{\mathrm{H}}+\frac{1}{2}\left\langle\begin{bmatrix}\hat{N}\\\hat{M}\end{bmatrix},\begin{bmatrix}\hat{\eta}\\\hat{\mathrm{N}}\end{bmatrix}\right\rangle_{\partial\mathrm{G}}+\left\langle\begin{bmatrix}C\eta\\EI\mathrm{N}\end{bmatrix},\begin{bmatrix}\dfrac{\partial u}{\partial x}+\dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2\\-w''\end{bmatrix}-\begin{bmatrix}\hat{\eta}\\\hat{\mathrm{N}}\end{bmatrix}\right\rangle_{\partial\mathrm{G}}$$

$$-\left\langle\begin{bmatrix}0\\\hat{Q}\end{bmatrix},\begin{bmatrix}u\\w\end{bmatrix}\right\rangle_{\partial\mathrm{H}}$$

APPLICATION OF THE QUADRATIC FUNCTIONAL: THE BLASIUS PROBLEM

Introduction

When the laminar boundary layer on a flat plate parallel to a stream is analysed, the following boundary-value problem, known as Blasius problem, arises:

$$f'''(\eta)+f(\eta)f''(\eta)=0 \quad , \quad \eta>0$$

with its boundary conditions:

$$f(0)=f'(0)=0$$

$$\lim_{\eta\to\infty} f'(\eta)=2$$

This problem, can be transformed, by a suitable substitution of variables

$$u=\frac{1}{2}f''(\eta)$$

$$x=\frac{1}{2}f'(\eta)$$

into the system:

$$\begin{aligned} &u(x)u''(x)+2x=0 \quad , \quad 0<x<1\\ &u'(0)=0 \qquad\qquad u(1)=0 \end{aligned} \tag{3}$$

Associated functional

A quadratic functional can be associated to the Blasius problem:

$$g(u)=\frac{1}{2}\int_0^1 (uu''+2x)^2\,dx \tag{4}$$

Its variation produces:

$$\int_0^1 (uu''+2x)(u''\delta u+u\delta u'')dx$$

which, under the hypothesis that the equation (11.1.1) has solution, as demonstrated in ALTMAN & OLIVEIRA (1982), presents, for the solution of the given problem, the varied form of the quadratic functional

$$\int_0^1 (uu''+2x)(u''\delta u+u\delta u'')dx=0 \tag{5}$$

Approximations for u(x)

Consider, taking into account the boundary and differentiation conditions established for $u(x)$, the following polynomial approximations:

$$u = a\xi + b\xi^2$$
$$u = a\xi + b\xi^2 + c\xi^3$$
$$u = a\xi + b\xi^2 + c\xi^3 + d\xi^4$$

where $\xi = 1 - x^3$ with a, b, c and d are constants to determine.

Each one of these approximations was used to solve the problem formulated through the equation (5). The constants a, b, c and d, were determined in each case using the Newton's iterative method, according to BAKHALOV (1977), to solve the corresponding system of algebraic non-linear equations.

Numerical results

It is presented in the table 4.1, at typical abscissas of the interval [0,1], the values for $u(x)$ obtained by the application of the equation resulting of the variation of the functional (4), closing the table with the reference solution extracted from CALLEGARI & FRIEDMAN (1968).

In this table the three lines corresponding to the mentioned equation refer, respectively, to the polynomial approximations of second, third and fourth degrees shown in equations (6)-(8).

Table 11.1 Exact and approximated values for u(x)

	0.00000	0.20000	0.40000	0.60000	0.80000	1.0
(6)	0.62727	0.62440	0.60333	0.53738	0.37369	0.0
(7)	0.67354	0.66871	0.63551	0.54950	0.38073	0.0
(8)	0.65388	0.65052	0.62550	0.54881	0.37789	0.0
(30)	0.66412	0.66009	0.63167	0.55183	0.38189	0.0

Resumo

Para sistemas mecânicos não conservativos (operadores não potenciais), os princípios variacionais não se aplicam. Neste artigo, um princípio variacional generalizado, válido para sistemass não lineares e não conservativos é deduzido por meio de operadores potencializáveis. Uma inclusão sistemática das condições de contorno em problemas lidando com tais operadores é proposta e exemplos extraídos da mecânica do contínuo são apresentados.

Palavras-Chave: Princípios variacionais, mecânica não linear, sistemas mecânicos não conservativos

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